International Journal of Theoretical Physics, Vol. 36, No. 12, 1997

Analytic Solutions of the Schrödinger Equation for the Modified Quartic Oscillator

L. Skála,[1] J. Dvořák,[1] and V. Kapsa[1]

Received June 19, 1997

No analytic solutions of the Schrödinger equation are known for the quartic anharmonic oscillator. We show in this paper that there are closely related modified quartic oscillators with the potential depending on $|x|$ for which analytic solutions for some states exist. These results can be extended to the higher order oscillators.

1. INTRODUCTION

Analytic solutions of the Schrödinger equation for the anharmonic oscillators

$$\left(-\frac{d^2}{dx^2} + V\right)\psi = E\psi \tag{1}$$

where $V = x^{2N}$ and $N = 2, 3, \ldots$, are not known (see, e.g., Bender *et al.*, 1969; Bender and Wu, 1971, 1973; Bender, 1982; Simon, 1970, 1982; Fernandez *et al.*, 1985; Killingbeck *et al.*, 1985; Richardson and Blankenbecler, 1979; Weniger *et al.*, 1991, 1993; Vinette and Čížek, 1991). However, it has been shown that analytic solutions exist for some polynomial potentials of the order $2N = 4k + 2$, where $k = 1, 2, \ldots$ (see, e.g., Magyari, 1981; Turbiner and Ushveridze, 1987; Ushveridze, 1994; Vanden Berghe *et al.*, 1995; Skala *et al.*, 1996). These solutions have the form

$$\psi = P(x)e^{Q(x)} \tag{2}$$

where $P(x)$ and $Q(x)$ are polynomials. Such analytic solutions exist for some polynomial potentials in which $(N - 1)$ constraints on the values

[1] Faculty of Mathematics and Physics, Charles University, 121 16 Prague 2, Czech Republic; e-mail: skala@quantum.karlov.mff.cuni.cz.

0020-7748/97/1200-2953$12.50/0

of the potential coefficients are introduced (Magyari, 1981; Turbiner and Ushveridze, 1987; Ushveridze, 1994; Vanden Berghe *et al.*, 1995; Skála *et al.*, 1996). We note that there are no constraints for the harmonic oscillator ($N = 1$) only. It appears that the potential constraints depend on the order of the polynomial $P(x)$. For this reason, there is only a finite number of the analytic solutions of the form (2) for given potential constraints, and different analytic solutions correspond usually to different potential constraints. The harmonic oscillator with no potential constraints is a special case in which all analytic solutions belong to the same potential.

Assuming the polynomial potential of the order $2N$, the solutions of the differential equation (1) behave for $x \to \pm\infty$ as $\exp[\pm x^{N+1}/(N+1)]$. To get the square-integrable wave function ψ for $2N = 4k + 2$ and $k = 0, 1, \ldots,$ the highest order term in the polynomial $Q(x)$ must be taken in the form $-x^{2k+2}/(2k+2)$. For $2N = 4k$, $k = 1, 2, \ldots,$ the highest order term in $Q(x)$ must be taken in the form $-x^{2k+1}/(2k+1)$ for $x > 0$ and $x^{2k+1}/(2k+1)$ for $x < 0$ or, equivalently, in the form $-|x|^{2k+1}/(2k+1)$. We see that to find the analytic solutions in this case we have to solve $x > 0$ and $x < 0$ cases separately and match the solutions at $x = 0$. In comparison with the $2N = 4k + 2$ case, there are two additional conditions here. First, both solutions for $x > 0$ and $x < 0$ must correspond to the same eigenvalue E and potential V. Second, the wave function and its first derivative must be continuous at $x = 0$. These conditions limit our possibilities to find the analytic solutions in the $2N = 4k$ case. This is obviously the reason that no analytic solutions of this problem have been found till now.

In this paper, we investigate the problem of the generalized quartic oscillator with the potential

$$V = V_1 x + V_2 x^2 + V_3 x^3 + x^4 \tag{3}$$

or its modifications. In Section 2, we summarize the matrix representation introduced in Skála *et al.*, (1996). In the following section, we show that the analytic ground-state wave function exists for the modified quartic oscillator with the potential $V = -2|x| + V_3^2 x^2/4 + V_3|x|^3 + x^4$ (Section 3). Excited states for the potential $V = -2(n+1)|x| + x^4$ are found in Section 4. In Section 5, we discuss excited states for some more general potentials. Conclusions and discussion of some higher order oscillators are given in the last section.

2. MATRIX REPRESENTATION

First we transform the Schrödinger equation (1) with the potential (3) into the matrix form. Following the general method suggested in Skála *et al.*, (1996), we assume the wave function ψ in the form

$$\psi = \sum_{m=0}^{\infty} c_m x^m e^{-g_0 x - g_1 x^2/2 - g_2 x^3/3} \tag{4}$$

where c_m and g_m are coefficients of the polynomials $P(x)$ and $Q(x)$. For the sake of simplicity, the wave functions investigated in this paper are not normalized.

It has been shown in Skála *et al.*, (1996) that

$$H\psi_m = \sum_k h_{mk}\psi_k \tag{5}$$

where

$$\psi_m = x^m e^{-g_0 x - g_1 x^2/2 - g_2 x^3/3}$$

and coefficients h_{mk} equal

$$\begin{aligned} h_{m,m+i} = & -m(m-1)\delta_{i,-2} + 2mg_0\delta_{i,-1} + (2mg_1 + g_1 - g_0^2)\delta_{i,0} \\ & + (2mg_2 - 2g_0g_1 + 2g_2 + V_1)\delta_{i,1} + (-2g_2g_0 - g_1^2 + V_2)\delta_{i,2} \\ & + (-2g_1g_2 + V_3)\delta_{i,3} + (-g_2^2 + 1)\delta_{i,4} \end{aligned}$$

Substituting (4) and (5) into the Schrödinger equation (1) and assuming the linear independence of the functions ψ_m, we get the equivalent matrix eigenvalue problem of the infinite order

$$\sum_m c_m h_{mk} = Ec_k \tag{6}$$

The coefficients c_m are the components of the left eigenvector of the matrix $h = \{h_{mn}\}$. The matrix h is not Hermitian.

3. GROUND STATE

It is well known that the wave function which is the solution of the one-dimensional Schrödinger equation with the energy $E_0 < E_i < E_2 \ldots$ has $n = 0, 1, 2, \ldots$ nodes.

To calculate the ground-state wave function with no nodes, we assume the function in the form

$$\psi(x) = e^{-g_0 x - g_1 x^2/2 - g_2 x^3/3} \tag{7}$$

To obey the matrix problem (6) for $c_m = \delta_{m0}$ and the boundary condition $\psi(x) \to 0$ for $x \to \infty$ we take

$$g_2 = 1, \qquad g_1 = V_3/(2g_2), \qquad g_0 = (V_2 - g_1^2)/(2g_2) \tag{8}$$

and introduce the potential constraint

$$V_1 = 2g_1g_0 - 2g_2 \tag{9}$$

It is obvious that the function (7) is the solution of the differential equation (1) for $E = g_1 - g_0^2$. However, it does not obey the boundary condition $\psi(x) \to 0$ for $x \to -\infty$. Therefore, it is the solution for $x > 0$ only. To get the solution for $x < 0$ the opposite sign $g_2 = -1$ must be taken.

Changing the sign of g_2, the coefficients g_1 and g_0 in (8) change their sign as well. Therefore, to get the same energy E for both solutions $x > 0$ and $x < 0$ we have to assume that V_3 changes the sign, too. In this case, g_1 does not change the sign and the right-hand side of (9) changes sign. Therefore, V_1 must change the sign similarly to V_3.

The derivative of the wave function (7) must be continuous at $x = 0$. It leads to $V_2 = V_3^2/4$ and $g_0 = 0$.

As a result, we get the analytic ground-state wave function for the potential

$$V = -2|x| + V_3^2x^2/4 + V_3|x|^3 + x^4$$

where V_3 can have an arbitrary value. The ground-state wave function for this potential equals

$$\psi(x) = e^{-V_3x^2/4-|x|^3/3}$$

and corresponds to the energy $E = V_3/2$.

4. POTENTIAL $V = -2(n + 1)|x| + x^4$, $n = 0, 1, \ldots$

The wave functions corresponding to this potential can be found in a similar way as the ground-state wave function. For the sake of simplicity, we assume in this section the potential in a special form

$$V = V_1|x| + x^4 \tag{10}$$

Similarly to the ground-state wave function, the wave functions depend on $|x|$.

For $x > 0$, we assume the wave function in the form (4). The infinite-order problem (6) can be reduced to a finite-order eigenvalue problem

$$\sum_{m=0}^{n} c_m h_{mk} = Ec_k, \qquad k = 0, \ldots, n \tag{11}$$

if the conditions $h_{n,n+i} = 0$, $i = 1, \ldots, 4$, are obeyed (Skála *et al.*, 1996). It leads to the equations $g_0 = g_1 = 0$, $g_2 = 1$, and the potential constraint

$$V_1 = -2(n + 1), \qquad n = 0, 1, \ldots. \tag{12}$$

The matrix h then equals

$$h_{m,m+i} = -m(m - 1)\delta_{i,-2} - 2(n - m)\delta_{i,1}$$

The nonzero coefficients c_m are given by the recurrence equation following from (11)

$$c_n = 1$$

$$c_{m-3} = -\frac{m(m - 1)}{2(n - m + 3)} c_m$$

The case $x < 0$ can be discussed analogously and gives similar results.

From the recurrence equations, it is easy to construct the wave functions. The analytic functions exist for $n = 0, 1, 3, 4, 6, 7, 9, 10, \ldots$. These functions are the solutions of the Schrödinger equation (1) with the potential given by (10) and (12) for the energy $E = 0$. The number i shown together with the function denotes the number of its nodes:

$$\psi_0(x) = e^{-|x|^3/3}, \qquad i = 0$$

$$\psi_1(x) = xe^{-|x|^3/3}, \qquad i = 1$$

$$\psi_3(x) = (-1 + |x|^3)e^{-|x|^3/3}, \qquad i = 2$$

$$\psi_4(x) = x(-2 + |x|^3)e^{-|x|^3/3}, \qquad i = 3$$

$$\psi_6(x) = (5/2 - 5|x|^3 + x^6)e^{-|x|^3/3}, \qquad i = 4$$

$$\psi_7(x) = x(7 - 7|x|^3 + x^6)e^{-|x|^3/3}, \qquad i = 5$$

$$\psi_9(x) = (-10 + 30|x|^3 - 12x^6 + |x|^9)e^{-|x|^3/3}, \qquad i = 6$$

$$\psi_{10}(x) = x(-35 + 105|x|^3/2 - 15x^6 + |x|^9)e^{-|x|^3/3}, \qquad i = 7$$

The function $\psi_0(x) = \exp(-|x|^3/3)$ is identical with the wave function following from the asymptotic analysis for $x \to \pm\infty$ and is a special case of the ground-state wave function found in the preceding section.

To clarify the role of V_1 in the potential (12), we assume that $V_1 < 0$. This potential has a double-well form with the maximum at $x = 0$. For $V_1 = -2$, the ground-state wave function ψ_0 corresponds to the energy $E = 0$ lying at the potential maximum $V(0) = 0$. Increasing V_1, the depth of the wells increases and the energy levels move down. For $V_1 = -4$, the second level goes through the potential maximum and the corresponding analytic wave function is ψ_1. Increasing n, this situation repeats and the higher excited-state wave functions corresponding to $E = 0$ are successively obtained. It is obvious that for $V_1 = -2(n + 1)$, $n = 0, 1, \ldots$, all the analytic states of this kind are obtained.

The functions $\psi_n(x)$, $n = 0, 1, \ldots$, have an interesting property. It follows from the equation

$$\left(-\frac{d^2}{dx^2} + x^4\right)\psi_n(x) = 2(n + 1)|x|\psi_n(x)$$

that the functions $\psi_n(x)$ are orthogonal if the weight $|x|$ in the integration is used.

5. EXCITED STATES FOR MORE GENERAL POTENTIALS

General discussion of the problem (6) is difficult. For this reason, we discuss only a few cases in which this problem can be reduced to the diagonalization of the problem (11). The case $n = 0$ has been discussed in Section 3.

Case n = 1

The analysis of (6) shows that it can be reduced to the problem (11) of the order two in the following cases.

In the first case, the preexponential factor equals x. We find the analytic wave function in the form

$$\psi(x) = xe^{-V_3x^2/4-|x|^3/3}$$

This function describes the first excited state of the Schrödinger equation with the potential

$$V(x) = -4|x| + V_3^2x^2/4 + V_3|x|^3 + x^4$$

for the energy $E = 3V_3/2$. In the special case $V_3 = 0$ the solution $\psi_1(x)$ from Section 4 is obtained.

In the second case, the preexponential factor is a first-order polynomial in $|x|$. The wave function with the preexponential factor equal to $|x|$ does not have a continuous derivative at $x = 0$. Therefore, we have to assume $c_0 \neq 0$. We find the wave function in the form

$$\psi(x) = (c_0 + |x|)e^{-(V_2-V_3^2/4)|x|/2-V_3x^2/4-|x|^3/3}$$

This function is the solution of the Schrödinger equation for the potential

$$V = [(V_2 - V_3^2/4)V_3/2 - 4]|x| + V_2x^2 + V_3|x|^3/3 + x^4$$

The energy E depends on V_2 and V_3 and can have two values

$$E^{\pm} = -V_2^2/4 - V_3^4/64 + V_3 + V_2V_3^2/8 \pm 1/2\sqrt{3V_3^2 - 8V_2}$$

The corresponding values of c_0 are

$$c_0^{\pm} = V_3/4 \mp 1/4\sqrt{3V_3^2 - 8V_2}$$

Depending on the sign of c_0, the ground state ($c_0 > 0$) or the first excited state ($c_0 < 0$) is obtained. The condition of the continuous derivative of the wave function at $x = 0$ leads to the potential constraint between V_2 and V_3,

$$V_2^{\pm} = 3V_3^2/8 - (A^{\pm})^2/8$$

where $A^{\pm}$ denotes the real root of the polynomial

$$(A^{\pm})^3 \mp V_3(A^{\pm})^2 - V_3^2A^{\pm} \pm V_3^3 \mp 64$$

Depending on the value of V_3, different numbers of physically relevant solutions can exist.

In the special case $V_3 = 0$ the wave function

$$\psi(x) = (-1 + |x|)e^{|x|-|x|^3/3}$$

is obtained. This function is the solution of the Schrödinger equation with the potential

$$V = -4|x| - 2x^2 + x^4$$

corresponding to the second excited-state energy $E = 1$.

Case $n = 2$

In this case, the eigenvalue problem (11) is of the order three. For the sake of simplicity, we assume $V_3 = 0$. Similarly as for $n = 1$ we get the analytic solution

$$\psi(x) = (c_0 + c_1|x| + c_2x^2)e^{-V_2|x|/2-|x|^3/3} \tag{13}$$

where

$$c_0 = (16E^2 + 8EV_2^2 + V_2^4 + 64V_2)/128$$

$$c_1 = -(E/2 + V_2^2/8)$$

$$c_2 = 1$$

and

$$E = A^{1/3} - 8V_2/(3A^{1/3}) - V_2^2/4, \qquad A = 8(\sqrt{24V_2^3/81 + 1} - 1)$$

The value of the potential coefficient V_2 is given by the equation

$$E^2V_2/8 + E(1 + V_2^3/16) + V_2^5/128 + 3V_2^2/4 = 0$$

which can be solved only numerically. We obtained $V_2 = 1.399086777$. The wave function (13) is the ground-state wave function for the potential

$$V = -6|x| + V_2x^2 + x^4$$

Case $n \geq 3$

For $n = 3$, the eigenvalue problem (11) can be solved in terms of Cardan's formulas. One of the wave functions is $\psi_3(x)$ given is Section 4. The calculation of other functions is too cumbersome and will not be given here.

For $n > 3$ and $V = -2(n + 1)|x| + x^4$, we get the functions $\psi_n(x)$ found in Section 4. Other states can be calculated numerically from (11). Following Ushveridze (1994), the corresponding solutions can be denoted as quasi-exact solutions.

6. CONCLUSIONS

To find the analytic solutions for the modified quartic oscillator, the general method suggested in Skála et al. (1996) has been used. We have shown that to get the analytic solutions for this oscillator it is necessary to introduce the absolute value $|x|$ into the potential. Similarly to other analytically solvable cases, the potential coefficients cannot be arbitrary and must obey certain potential constraints. In detail, we have discussed the analytic solutions for the ground state and few lowest excited states. For the special form of the potential $V = -2(n + 1)|x| + x^4$, the analytic ground-state as well as excited-state wave functions have been found. More general cases lead to the eigenvalue problem (11), which can be solved numerically.

The most interesting application of these results seems to be in the theory of the quartic oscillator with the potential $V = x^2 + \beta x^4$. The usual perturbation theory for this oscillator is based on the use of the zero-order Hamiltonian $H_0 = x^2$ (see, e.g., Bender, *et al.*, 1969; Bender and Wu, 1971, 1973; Bender, 1982; Simon, 1970, 1982). The well-known divergence of the corresponding perturbation series is related to different asymptotic behavior of the wave functions of the harmonic and quartic oscillator for $x \to \pm\infty$ (Weniger, 1996). From this point of view, the analytic wave functions found in Section 4 are more suitable for the use in the perturbation theory than the functions of the harmonic oscillator. We would like to investigate this problem not only for the quartic, but also for the sextic oscillator and higher order oscillators.

It is obvious that the results obtained for the modified quartic oscillator can be extended to higher order oscillators. For example, we can consider the potential

$$V = -N|x|^{N-1} + x^{2N}$$

where $N = 1, 2, 3, 4, \ldots$. Analogously to the discussion in Section 4, it can be shown that the ground-state wave function for this potential equals

$$\psi_0(x) = e^{-|x|^{N+1}/(N+1)}$$

Similary to functions $\psi_n(x)$ from Section 4, this wave function obeys the Schrödinger equation (1) for $E = 0$. This solution can be also obtained from the general confluent equation (Abramowitz and Stegun, 1972) or from the differential equation $y'' + (ax^{c-2} - b^2x^{2c-2})y = 0$ (Kamke, 1956) in terms of the Whittaker functions $M_{\kappa,\mu}$ and $W_{\kappa,\mu}$. However, our approach is more simple and straightforward and makes it possible to find analytic solutions even for $E \neq 0$. These examples show that the analytic solutions of the Schrödinger equation exist not only for some polynomial potentials of the order $2N = 4k + 2$, but also for some polynomial potentials of the order $2N = 4k$, provided that the variable x is replaced by $|x|$.

REFERENCES

Abramowitz, M., and Stegun, I. A., eds. (1972). *Handbook of Mathematical Functions*, Dover, New York, paragraph 13.1.35.

Bender, C. M. (1982). *International Journal of Quantum Chemistry*, **21**, 93.

Bender, C. M., and Wu, T. T. (1969). *Physical Review*, **184**, 1231.

Bender C. M., and Wu, T. T. (1971). *Physical Review Letters*, **27**, 461.

Bender, C. M., and Wu, T. T. (1973). *Physical Review D*, **7**, 1620.

Fernández, F. M., Mesón, A. M., and Castro, E. A. (1985). *Journal of Physics A*, **18**, 1389.

Kamke, E. (1956). *Differentialgleichungen. Lösungmethoden und Lösungen, Akademische Verlagsgesellschaft*, Leipzig, Vol. I, paragraph 2.273, Eq. (12).

Killingbeck, J., Jones, M. N., and Thompson, M. J., (1985). *Journal of Physics A*, **18**, 793.

Magyari, E. (1981). *Physics Letters A*, **81**, 116.

Richardson, J. L., and Blankenbecler, R. (1979). *Physical Review D*, **19**, 496.

Simon B. (1970). *Annals of Physics*, **58**, 76, Simon, B. (1982). *International Journal of Quantum Chemistry*, **21**, 3.

Skála, L., Dvořák, J., Čížek, J., Špirko, V. (1996). *Physical Review A*, **53**, 2009.

Turbiner, A. V., and Ushveridze, A. G. (1987) *Physics Letters A*, **126**, 181.

Ushveridze, A. G. (1994). *Quasi-Exactly Solvable Models in Quantum Mechanics*, Institute of Physics Publishing, Bristol.

Vanden Berghe, G., De Meyer, H., and Van Daele, M. (1995). *International Journal of Theoretical Physics*, **34**, 1931.

Vinette, F., and Čížek, J. (1991). *Journal of Mathematical Physics*, **32**, 3392.

Weniger, E. J. (1996). *Annals of Physics*, **246**, 133.

Weniger, E. J., Čížek, J., and Vinette, F. (1991). *Physics Letters A*, **156**, 169.

Weniger, E. J. Čížek, J., and Vinette, F. (1993). *Journal of Mathematical Physics*, **34**, 571.